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Singular Perturbation of Linear Partial Differential Equation

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I

The aim of this paper is to show the convergence of certain solution of the homogeneous linear partial differential equation

$$L_\epsilon(u) = \epsilon\, \delta_t^2 u + P_1(\delta_x, t)\, \delta_t u + P_2(\delta_x, t, \epsilon) u = 0 \tag{1}$$

as $\epsilon > 0$ goes to zero[1] to a solution of the limit equation ($\epsilon = 0$)

$$L_0(u) = 0 \tag{1^0}$$

where $P_i(\delta_x, t, \epsilon)$ $(i = 1, 2)$ are polynomials in $\delta_{x_1}, \delta x_2, \ldots, \delta_{x_m}$ and ϵ with complex coefficients which are C^∞ functions in the real parameter t in $J = [0, T]$ for some fixed positive number T.

We demand here that $P_1(\delta_x, t)$ does not depend on ϵ.

This relationship has been discussed in [*1*] for equation of order n but with constant coefficients. Here we permit the coefficients to depend on the real parameter t.

[Here I wish to thank the referee for his suggestion which enables me to reduce the size of this paper considerbly.]

DEFINITION (1). For any integer $p > 0$ and any function $u(x)$ in C_0^∞ let the norm of u be defined as follows:

$$\| u \|_p^2 = \int_{E^m} |(1 + | z |^2)^{p/2}\, \hat{u}(z)|^2\, dz \tag{2}$$

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[1] In what follows there will be for each ϵ certain exceptional sets of z (z in E^m) of measure zero for which our conclusions do not apply. In order to be able to draw inferences as $\epsilon \to 0$, $\epsilon > 0$ we wish to be able to disregard these sets.

Now let the notion $\epsilon \to 0$ be henceforth interpreted as meaning "ϵ tends to zero through an arbitrary sequence of positive numbers." Then all of the corresponding exceptional sets will still be according to Theorem (2) in [*1*] a countable union of sets of measure zero and accordingly itself has measure zero.

($\wedge$ is the Fourier transformation). The space C_0^∞ with the norm (3) gives a Hilbert space. We shall call it $H_{p,x}$ space.

DEFINITION (2). Let $u(t)$ be a variable element of $H_{p,x}$ depending a real parameter t in J. Then $u(t)$ is $H_{p,x}$ continuous in t if the mapping t in $J \to u(t)$ in $H_{p,x}$ is continuous, i.e. $t \to t_0$ in J implies that $u(t) \to u(t_0)$ in $H_{p,x}$. We say that $u(t)$ is $H_{p,x}$-differentiable at $t = t_0$, if there exists a function $g(t)$ in $H_{p,x}$ such that $(t - t_0)^{-1}[u(t) - u(t_0)] \to g(t_0)$ in $H_{p,x}$ as $t \to t_0$. Then we denote $g(t)$ by $(d/dt)\, u(t)$.

DEFINITION (3). We say that equation (1) is $H_{p,x}$-stable equation for $\epsilon \to 0$ in $0 \leqslant t \leqslant T$ with respect to a particular solution $u(t)$ of (1^0) for ($\epsilon = 0$), if and only if, $u_\epsilon(t) \to u_0(t)$ in $H_{p,x}$ for t in J whenever $u_\epsilon(t, x)$ is an $H_{p,x}$ solution of equation (1) with the property:

$$u_\epsilon(0) \to u_0(0) \tag{3}$$

in $H_{p,x}$ and there exists a function $F(x)$ in $H_{p,x}$ with the following property:

$$|\, \delta_t \hat{u}(0, z)| \leqslant |\, \hat{F}(z)| \tag{4}$$

for all small $\epsilon > 0$.

II

In this section we shall derive an asymptotic expansion for $Y_j(t, z, \epsilon)$ (the solution of equation (2)) with the initial conditions

$$\delta_t^{k-1} Y_j(0, z, \epsilon) = \delta_{jk}\,. \tag{5}$$

Just for the simplicity, we made the order of equation (1) be 2. Otherwise it would be harder to compute the constants in $Y_j(t, z, \epsilon)$ by using the above initial conditions.

As in [1] we associate equation (1) with the linear ordinary differential equation

$$\epsilon D_t^2 u + P_1(iz, t)\, D_t u + P_2(iz, t, \epsilon) u = 0 \tag{6}$$

for z in E^m.

LEMMA (1). *Let $Y_j(t, z, \epsilon)$ be the solutions of equation* (6) *with the initial conditions* (5) *satisfying the following condition: There exist two constants $\epsilon_0 > 0$ and C such that*

$$\operatorname{Sup} |\, Y_j(t, z, \epsilon)| \leqslant C \tag{7}$$

for all t in J and $0 < \epsilon \leqslant \epsilon_0$ *and the* sup *is taken over all z in* E^m. *Then real* $[P_1(iz, t)]$ *is nonnegative for all t in J.*

Proof. We rewrite equation (6) as

$$D[\epsilon Dy + P_1(iz, t)y] + \tilde{P}_2(iz, t, \epsilon)y = 0$$

where; $\tilde{P}_2 = (dP_1/dt) + P_2$. Finding u from the equation

$$D_t u + \tilde{P}_2(iz, t, \epsilon)y = 0$$

and substituting it in the equation

$$\epsilon D_t y + P_1(iz, t)y = u$$

we find that

$$y = \exp\left[-(1/\epsilon)\int P_1(iz, t)\,dt\right]$$
$$\times\left[\int^t \exp\left[(1/\epsilon)\int^s P_1(iz, t)\,dt\right]\frac{1}{\epsilon}\left(\int^s -\tilde{P}_2(iz, t, \epsilon)\,y\,dt + C_2\right)ds + C_1\right]$$

For $y = Y_1(iz, t, \epsilon)$ using the initial conditions (5) we find $C_2 = 0$ and $C_1 = 1$. Consequently

$$Y_1(iz, t, \epsilon) = [1 + G(z, t, \epsilon)]\exp\left[-(1/\epsilon)\int P_1(iz, t)\,dt\right]$$

where

$$G(z, t, \epsilon) = \int^t \exp\left[(1/\epsilon)\int^s P_1(iz, t)\,dt\right]\frac{1}{\epsilon}\left(\int^s -\tilde{P}_2(iz, t, \epsilon)\,Y_1\,dt\right)ds.$$

Using the triangular inequality we get

$$|\,Y_1(t, z, \epsilon)| \geqslant \left|\exp\left[-\frac{1}{\epsilon}\int P_1(iz, t)\,dt\right]\right|\,|\,1 - |\,G(z, t)||\,. \qquad (*)$$

Notice that $Y_1(t, z\epsilon)$ is bounded and $\tilde{P}_2(iz, t, \epsilon)$ is a polynomial in ϵ, therefore if, real $P_1(iz, t) < 0$ then Ascoli's theorem implies that

$$|\,G(z, t, \epsilon)| \to 0$$

as $\epsilon \to 0$. The inequality (*) shows that

$$|\,Y_1(t, z, \epsilon)| \to \infty$$

as $\epsilon \to 0$ and this is a contradiction. The lemma is proved.

Let $u^{[k]} = \epsilon^k\, d^k u/dt^k$. Inserting this in equation (6), we have

$$Ay \equiv y^{[2]} + P_1(iz, t)\,y^{[1]} + \epsilon P_2(iz, t, \epsilon)\,y = 0. \qquad (8)$$

In an expression like $e^{(1/\epsilon)f(t)}(e_0(t)\,\epsilon^k + e_1(t)\,\epsilon^{k+1} + \cdots)$, the term $e_0(t)^k$ shall be called the principal term.

We observe that the roots of the polynomial equation

$$w^2 + P_1(iz, t)\, w = 0 \tag{9}$$

are distinct except for a set of E^m measure zero. They are

$$w_1 = 0,$$

and

$$w_2 = -P_1(iz, t).$$

Hence by Lemma (1) we conclude the following:

$$| e^{\frac{1}{\epsilon} w_2} | < | e^{\frac{1}{\epsilon} w_1} | = 1. \tag{9'}$$

LEMMA (2). *For every value of i ($i = 1, 2$) there exists an infinite number of functions $u_{i0}(t), u_{i0}(t),\ldots$, continuous and with continuous derivatives of all orders such that $u_{i0}(t)$ does not vanish at any point of J and if the functions*

$$u_i(t, \epsilon) = \exp\left[\frac{1}{\epsilon}\int_0^t w_i(s)\, ds\right] \sum_{j=0}^{m-1} u_{ij}(t)\, \epsilon^j$$

are substituted in the expression $A(y)$ for y, then the coefficient of

$$\exp\left[\frac{1}{\epsilon}\int_0^t w_i(s)\, ds\right] \epsilon^h \qquad \begin{matrix}(i = 1, 2)\\ (h = 0, 1, 2,\ldots, m)\end{matrix}$$

in the expansion thus obtained vanishes identically.

Proof. Write $y(t, \epsilon) = \exp[1/\epsilon \int_0^t w(s)\, ds]\, v(t, \epsilon)$ where $w(s)$ is some definite one of the two roots $w_i(t)$. We find the following:

$$y^{[1]}(t, \epsilon) = \exp\left[\frac{1}{\epsilon}\int_0^t w(s)\, ds\right][w(t)\, v(t, \epsilon) + {}^{[1]}(t, \epsilon)]$$

$$y^{[2]}(t, \epsilon) = \exp\left[\frac{1}{\epsilon}\int_0^t w(s)\, ds\right]((w(t)^2 + \epsilon(d/dt)\, w(t))\, v(t, \epsilon)$$
$$+ 2w(t)\, v^{[1]}(t, \epsilon) + v^{[2]}(t, \epsilon)).$$

Hence we conclude

$$A(y) = \exp\left[\frac{1}{\epsilon}\int_0^t w(s)\, ds\right] \bar{A}(v) \tag{10}$$

where

$$\bar{A}(V(t, \epsilon)) = v^{[2]} + a_1(t, \epsilon)\, v^{[1]} + a_0(t, \epsilon) v \tag{11}$$

with $a_1(t, \epsilon) = 2w(t) + P_1(iz, t)$,

$$a_0(t, \epsilon) = (w(t))^2 + \epsilon(d/dt)\, w(t) + P_1(iz, \epsilon)\, w(t) + P_2(iz, t, \epsilon).$$

For convenience we write:

$$a_2(t, \epsilon) = 1,$$
$$a_n(t, \epsilon) = 0. \qquad (n > 2).$$

If we let

$$a_n(t, \epsilon) \equiv \sum_{j=0}^{\infty} a_{nj}(t)\, \epsilon^j.$$

Then

$$a_{00}(t) = (w(t))^2 + P_1(iz, t)\, w(t) = 0 \tag{12}$$

$$a_{10}(t) = 2w(t) + P_1(iz, t) \tag{13}$$

so that $a_{10}(t) \neq pmJ$, the two roots $w_i(t)$ being distinct. If we then insert in (10)

$$v(t, \epsilon) = \sum_{j=0}^{m-1} u_j(t)\, \epsilon^j,$$

we find the condition that the coefficient of

$$\exp\left[\frac{1}{\epsilon}\int_0^t w(s)\, ds\right] \epsilon^h \qquad (h = 0, 1,..., m)$$

in $Ay(t, \epsilon)$ vanishes to be

$$\sum_{j+k+n=h} a_{ij}(t)(d^h/dt^h)\, y_k(t) = 0. \tag{14}$$

The equation (14) is true for $h = 0$ by (12). For the case $h \neq 0$ we can write (14) in the form:

$$a_{10}(t)(d/dt)\, u_{h-1}(t) + a_{01}(t)\, u_{h-1}(t)$$
$$+ \sum_{j+k+n=h}^{k<h-1} a_{ij}(t)(d/dt)^h\, u_k(t) = 0. \tag{15}$$

If in (14) $k = h$, then $n = j = 0$ and the term corresponding to this set of values has a coefficient $a_{00}(t) = 0$ if $k = h - 1$. Since this is the case, we have either $n = 1, j = 0$ or $n = 0, j = 1$, and the corresponding terms are the first terms of equation (15).

It appears then that $u_{h-1}(t)$ can be determined in term of u_{h-2}, u_{h-3},..., u_0 as a solution of certain linear differential equation of the first order which has no singular points in J, since $a_{10}(t) \neq 0$. Thus $u_0(t), u_1(t),..., u_{m-1}(t)$ can be obtained in succession from (14) for ($h = 1, 2,..., m$). For each $w_i(t)$ we obtain in this way a sequence of functions $u_{i0}(t), u_{i1}(t),...,$ such that if the expressions

$$u_i(t, \epsilon) = \exp\left[(1/\epsilon)\int_0^t w_i(s)\, ds\right] \sum_{j=0}^{m-1} u_{ij}(t)\, \epsilon^j$$

be substituted for y in $A(y)$ the coefficient of

$$\exp\left[(1/\epsilon)\int_0^t w_i(s)\, ds\right] \epsilon^h \qquad (i = 1, 2 \quad \text{and} \quad h = 0, 1,..., m)$$

vanishes by (15) since the conditions (14) are now satisfied for $h = 1, 2,..., m$. Furthermore the differential equation for $u_{i0}(t)$ is homogeneous, so that, by taking for $u_{i0}(t)$ a solution which is different from zero at one point of J, we are sure that $u_{i0}(t)$ does not vanish at any point of J. Since the $a_{ij}(t)$ were continuous, $u_{i0}(t), u_{i1}(t),...,$ are also continuous with all of their derivatives. The sequence of functions $u_{i0}(t), u_{i1}(t),...,$ has then the properties stated in the lemma.

LEMMA (3). *The homogeneous linear ordinary differential equation of order two with two solutions $u_i(t, \epsilon)$ has the form*

$$B(y) \equiv y^{[2]} + b_1(t, \epsilon)\, y^{[1]} + b_0(t, \epsilon)\, y = 0$$

for $\epsilon \leqslant \epsilon_0 > 0$.

$$|\, b_i(t, \epsilon)| \leqslant M : b_i(t, \epsilon) = \sum_{j=0}^{\infty} b_{ij}(t)\, \epsilon^j, \qquad (i = 1, 2). \tag{16}$$

The coefficients $b_{ij}(t)$ which appears here are continuous with all their derivatives, and

$$b_{10}(t) = P_1(iz, t) \qquad \text{and} \qquad b_{1j}(t) = 0 \tag{17}$$

for all $j > 0$; and if

$$P_2(iz, t, \epsilon) = \sum_0^{\infty} P_{2j}(t)\, \epsilon^j,$$

then

$$b_{0j} = P_{2j}(t), \qquad (j \leqslant m).$$

Notice that $b_{00}(t) = 0$.

Letting,

$$P_i(iz, t, \epsilon) = \sum_0^{\infty} P_{ij}(t)\, \epsilon^j,$$

then

$$b_{ij} = P_{ij}\,, \qquad (i = 1, 2).$$

Proof of Lemma (3). The homogeneous linear differential equation of order two with solutions $u_i(t, \epsilon)$ is

$$\begin{vmatrix} y^{[2]} & y^{[1]} & y \\ u_1^{[2]}(t, \epsilon) & u_1^{[1]}(t, \epsilon) & u_1(t, \epsilon) \\ u_2^{[2]}(t, \epsilon) & u_2^{[1]}(t, \epsilon) & u_2(t, \epsilon) \end{vmatrix} = 0. \tag{18}$$

For the element of this determinant we have

$$u_i^{[j]}(t, \epsilon) = \exp\left[(1/\epsilon)\int_0^t w_i(s)\,ds\right] \sum_{k=0}^{m+j-1} \lambda_{ijk}(t)\,\epsilon^k \tag{19}$$

where,

$$\lambda_{ij0}(t) = [w_i(t)]^j\, u_{i0}(t). \tag{20}$$

Thus if we factor out of (18)

$$\prod_{i=1}^{2} \exp\left[(1/\epsilon)\int_0^t w_i(s)\,ds\right]$$

the differential equation takes the form

$$\beta_2(t, \epsilon)\, y^{[2]} + \beta_1(t, \epsilon)\, y^{[1]} + \beta_0(t, \epsilon)\, y = 0 \tag{21}$$

where $\beta_i(t, \epsilon)$ are polynomials in ϵ. We have for the principal term of $\beta_2(t, \epsilon)$

$$\beta_{20}(t) = \begin{vmatrix} \lambda_{110} & \lambda_{100} \\ \lambda_{210} & \lambda_{200} \end{vmatrix}.$$

In view of (20) this last determinant may be written

$$\prod_{i=1}^{2} u_{i0}(t) \begin{vmatrix} w_1(t) & 1 \\ w_2(t) & 1 \end{vmatrix}.$$

This, except for a set of E^m measure zero, is not zero at any point on J as the $w_i(t)$ are distinct and the $u_{i0}(t) \neq 0$ by Lemma (2). Therefore for $\epsilon \leqslant \epsilon_0$

$$|\,\beta_i(t, \epsilon)/\beta_2(t, \epsilon)| \leqslant M : b_i(t, \epsilon) = |\,\beta_i(t, \epsilon)/\beta_2(t, \epsilon)| = \sum_{j=0}^{\infty} b_{ij}(t)\,\epsilon^j.$$

But we can write (21) in the form

$$y^{[2]} + b_1(t, \epsilon)\, y^{[1]} + b_0(t, \epsilon)\, y = 0. \tag{22}$$

Now the functions $b_{ij}(t)$ and their derivatives are continuous since the $w_i(t)$ and $u_i(t)$ are of this character. The first part of the lemma is thus proved.

Now let j_0 be the smallest value of j which, for some i and t,

$$b_{ij}(t) \neq P_{ij}(t)$$

From (19) and (20) we see that the principal term of

$$B[u_i(t, \epsilon) - A(u_i(t, \epsilon))] \doteq \overline{B - A}[u_i(t, \epsilon)] \tag{23}$$

is

$$\epsilon^{j_0} \left[\sum_{k=0}^{1} [b_{kj_0}(t) - P_{kj_0}(t)][w_i(t)]^k \right] u_{i0}(t). \tag{24}$$

Assume $j_0 \leqslant m$ if possible. In each part of the difference (23) the coefficient of

$$\exp \left[(1/\epsilon) \int_0^t w_i(s)\, ds\right] \epsilon^{j_0}$$

must then vanish, in the first since $u_i(t, \epsilon)$ are solutions of $B(y) = 0$; in the second part, by Lemma (2). Therefore from (24)

$$\sum_{k=0}^{1} [b_{kj_0}(t) - P_{kj_0}(t)][w_i(t)]^k = 0 \qquad (i = 1, 2)$$

we conclude that

$$b_{kj_0}(t) = P_{kj_0}(t), \qquad (k = 0, 1, 2).$$

The $w_i(t)$ being distinct. This is a contradiction. Hence $j_0 > m$. The proof of Lemma (3) is completed.

Consider the two equations

$$u_{tt} + P_1(t)\, u_t + (1/\epsilon)\, b_0 u = 0, \tag{25}$$

$$\begin{aligned} y_{tt} + P_1(t)\, y_t + (1/\epsilon)\, b_0 y &= [(1/\epsilon)\, b_0 - P_2] y \\ &= \epsilon^{m+1} Q(t, \epsilon) y \end{aligned} \tag{26}$$

where $Q(t, \epsilon)$ is continuous in t and ϵ.

Let $y = u(t, \epsilon)\, v(t, \epsilon)$ with $v(0, \epsilon) = 1$ and $v_t(0, \epsilon) = 0$ and $u(t, \epsilon) = u_i(t, \epsilon)$, $i = 1, 2$, given in Lemma (2). Substituting into (26) yields

$$\epsilon v_{tt} + (P_1 + 2\epsilon u_t/u)\, v_t = \epsilon^m Q v$$

which can be integrated to give

$$v(t) = 1 + \epsilon^m \int_0^t [u(s)]^{-2} \int_0^s Q(\sigma)\, v(\sigma)\, u^2(\sigma) \exp\left[-\frac{1}{\epsilon}\int_\sigma^s P_1(l)\, dl\right] d\sigma\, ds.$$

Upon multiplying by $u(t)$ and interchanging the order of integrations, the last equation becomes

$$y(t) = u(t) + \epsilon^m \int_0^t y(\sigma)\, K(t, \sigma, \epsilon)\, d\sigma \tag{27}$$

where

$$K(t, \sigma, \epsilon) = Q(\sigma) \int_\sigma^t [u(s)]^{-2}\, u(t)\, u(\sigma) \exp\left[-\frac{1}{\epsilon}\int_\sigma^s P_1(l)\, dl\right] ds$$

which implies that $K(t, \sigma, \epsilon)$ is bounded for $0 < \epsilon \leqslant \epsilon_0$ and for both $u = u_1$ or $u = u_2$.

The method of standard successive iterations shows that there exists a unique solution $y(t)$ to equation (27). Furthermore, the solution $y(t)$ is bounded for t in J and $0 \leqslant \epsilon \leqslant \epsilon_0$. As a consequence, we have

$$y_i(t) = u_i(t, \epsilon) + \epsilon^m E_0, \qquad \text{and} \quad \frac{\delta y_i}{\delta t} = \frac{\delta u_i}{\delta t} + \epsilon^{m-1} E_1$$

where E_0 and E_1 are functions in ϵ and others but bounded for $0 \leqslant \epsilon \leqslant \epsilon_0$. E_0 can be derived from equation (27) and E_1 can be found by taking the derivative of equation (27). We can sum the above results into the following theorem.

Theorem (1). *The ordinary differential equation* (8) *has two independent solutions* $y_i(t, z, \epsilon)$ $(i = 1, 2)$ *such that if the integer m is chosen at pleasure, then*

$$y_i(t, z, \epsilon) = u_i(t, \epsilon) + \epsilon^m E_0, \qquad \text{and} \quad \frac{\delta y_i}{\delta t} = \frac{\delta u_i(t, \epsilon)}{\delta t} + \epsilon^{m-1} E_1.$$

Now let y_1 and y_2 be as in Theorem (1). Therefore the general solutions of equation (2) are:

$$Y_1(t, z, \epsilon) = A_1 y_1 + A_2 y_2, \qquad Y_2(t, z, \epsilon) = B_1 y_1 + B_2 y_2.$$

Using the initial conditions (3) we compute the constants A_i and B_i as:

$$\begin{aligned} A_1 &= (-(d/dt(y_2(0, z, \epsilon))/H(z, \epsilon), \qquad A_2 = (d/dt)(y_1(0, z, \epsilon)/H(z, \epsilon) \\ B_1 &= (y_2(0, z, \epsilon))/H(z, \epsilon), \qquad B_2 = (-y_1(0, z, \epsilon))/H(z, \epsilon) \end{aligned} \tag{28}$$

where

$$H(z, \epsilon) = y_2(\epsilon)(d/dt)\, y_1(0, z\epsilon) - y_1(0, z, \epsilon)(d/dt)\, y_2(0, z, \epsilon).$$

By the fact that $w_1(t) = 0$, $w_2(t) = -P_1(iz, t)$ and by Theorem (1) we have

$$\begin{aligned} y_1(t, z, \epsilon) &= \sum_{j=0}^{m-1} u_{1j}(t)_{\epsilon^j} + \epsilon^m E_0 \\ y_2(t, z, \epsilon) &= \sum_{j=0}^{m-1} u_{2j}(t)_{\epsilon^j} \exp\left[\frac{-1}{\epsilon}\int_0^t P_1(iz,is)\, ds\right] + \epsilon^m E_0 . \end{aligned} \tag{29}$$

Lemma (2) indicates that for all t in J none of $u_{10}(t)$ and $u_{20}(t)$ is equal to zero. Therefore, except for a set of E^m measure zero, $H(z, \epsilon) \neq 0$ for all small $\epsilon \geqslant 0$. Consequently we have $B_1(z, \epsilon)$, $B_2(z, \epsilon)$, and $A_2(z, \epsilon)$ converging to zero as ϵ tends to zero (B_1, B_2, A_2 are converging as functions of ϵ; i.e. z is fixed.) Theorem (1) indicates that $y_i(t, z\epsilon)$ are bounded ($i = 1, 2$) in ϵ. Therefore we conclude that, except for a set of E^m measure zero, $Y_2(t, z, \epsilon)$ tends to zero as ϵ tends to zero for all t in J and each z in E^m, while $Y_1(t, z, \epsilon)$ goes to $A_1(z, 0)\, y_1(t, z, 0)$.

From (28) and (29) we can write

$$Y_1(t, z, \epsilon) = (u_{10}(t))/(u_{10}(0)) + o(\epsilon). \tag{30}$$

From equation (15) we can determine $u_{10}(t)$ as the solution of the equation

$$P_1(iz, t)\frac{du}{dt} + P_2(iz, t, 0) = 0. \tag{31}$$

Now (30) and (31) together show that

$$Y_1(iz, t, \epsilon) \to Y_1(iz, t, 0) \quad \text{as} \quad \epsilon \to 0$$

for all t in J, where $Y_1(t, z, 0)$ is the solution of the equation

$$P_1(iz, t)(d/dt)\, y + P_2(iz, t, 0) = 0 \tag{32}$$

with the initial condition $Y(0, z, 0) = 1$. Now we sum the above results in the following lemma.

LEMMA (4). *If $Y_j(t, z, \epsilon)$ are the solutions of equation* (6) *with the initial conditions* (5) *and satisfy the condition* (7), *then, except for a set of E^m measure zero, and for all z in E^m we have*

$$Y_1(t, z, \epsilon) \to Y_1(t, z, 0)$$

and

$$Y_2(t, z, \epsilon) \to 0$$

as ϵ tends to zero and for all t in J.

III

In this section we shall prove the main theorem of the paper.

THEOREM (2). *Let the degree of $P_j(iz, \epsilon, t)$ in z be at most k where* ($j = 1, 2$) *and for all t in J assume that $P_1(iz, t) \not\equiv 0$. Let also $u_0(t, x)$ be $H_{p+k,x}$, the continuously differentiable solution of the equation* (1^0). *Suppose that $Y_1(t, z, \epsilon)$ and $Y_2(t, z, \epsilon)$ satisfy condition* (7) *in Lemma* (1). *Then equation* (1) *is an H_p stable equation.*

Remark (1). If neither $P_1(iz, t)$ nor $P_2(iz, t, \epsilon)$ depends on t, then we can prove that condition (7) is equivalent to another condition on the roots of the polynomial equation

$$\epsilon w^2 + P_i(iz)\, w + P_2(iz, \epsilon) = 0.$$

Those who are interested may see a special case in [*3*].

Remark (2). Notice that if $P_1(iz, t) = 0$, we will lose both solutions of equation (6) as well as the initial conditions (5) by passing to the limit; so the condition that $P_1(iz, t)$ is not zero becomes necessary.

Proof of Theorem (2). As in Theorem (2) in [*1*], we can write

$$u(t, x, \epsilon) - u(t, x, 0) = (1/\sqrt{2\pi}^m) \int_{E^m} e^{ix\cdot z}(Y_1(t, z, \epsilon)\, \hat{u}(0, z, \epsilon)$$

$$- Y_1(t, z, 0)\, \hat{u}(0, z, 0))\, dz + (1/\sqrt{2\pi}^m) \int_{E^m} e^{ix\cdot z} Y_2(t, z, \epsilon)\, \delta_t\, \hat{u}(0, z, \epsilon)\, dz.$$

Using Fourier transform notations, we have

$$\begin{aligned} u(t, x, \epsilon) - u(t, x, 0) &= \mathscr{F}^{-1}(Y_1(t, z, \epsilon)\, \hat{u}(0, z, \epsilon) - Y_1(t, z, 0)\, \hat{u}(0, z, 0)) \\ &\quad + \mathscr{F}^{-1}(Y_2(t, z, \epsilon)\, \delta_t \hat{u}(0, z,)). \end{aligned}$$

Hence

$$\|u(t, x, \epsilon) - u(t, x, 0)\|_p = \left(\int_{E^m} |(1 + |z|^2)^{p/2} (\hat{u}(t, z, \epsilon) - \hat{u}(t, z, 0)|^2\, dz\right)^{1/2}$$

$$\leqslant \left(\int_{E^m} |(1 + |z|^2)^{p/2} (Y_1(t, z, \epsilon)\, \hat{u}(0, z, \epsilon)) - Y_1(t, z, 0)\, \hat{u}(0, z, 0))|^2\, dz\right)^{1/2}$$

$$+ \left(\int_{E^m} |(1 + |z|^2)^{p/2}\, Y_2(t, z, \epsilon)\, \delta_t \hat{u}(0, z, \epsilon)|^2\, dz\right)$$

$$\leqslant \left(\int_{E^m} |(1 + |z|^2)^{p/2} ((Y_1(t, z, \epsilon)\, \hat{u}(0, z, \epsilon) - \hat{u}(0, z0))\right.$$

$$\left. + (Y_1(t, z, \epsilon) - Y_1(t, z, 0))\, \hat{u}(0, z, 0)|^2\, dz\right)^{1/2}$$

$$+ \left(\int_{E^m} |(1 + |z|^2)^{p/2}\, Y_2(t, z, \epsilon)\, \delta_t \hat{u}(0, z, \epsilon)|^2\, dz\right)^{1/2}$$

$$\leqslant \left(\int_{E^m} |(1+|z|^2)^{p/2}\,(Y_1(t,z,\epsilon)[\hat{u}(0,z,\epsilon)-\hat{u}(0,z,0)])|^2\,dz\right)^{1/2}$$
$$+\left(\int_{E^m} |(1+|z|^2)^{p/2}\,(Y_1(t,z,\epsilon)-Y_1(t,z,0))\,\hat{u}(0,z,0)|^2\right)^{1/2}$$
$$+\left(\int_{E^m} |(1+|z|^2)^{p/2}\,Y_2(t,z,\epsilon)\,\delta_t\hat{u}(0,z,\epsilon)|^2\,dz\right)^{1/2}.$$

By using condition (7), we get

$$\int_{E^m} |(1+|z|^2)^{p/2}\,Y_1(t,z,\epsilon)[(\hat{u}(0,z,\epsilon)-\hat{u}(0,z,0)]|^2\,dz$$
$$\leqslant C\int_{E^m} |(1+|z|^2)^{p/2}\,(\hat{u}(0,z,\epsilon)-\hat{u}(0,z,0))|^2\,dz$$

and by condition (3) it follows that

$$\int_{E^m} |(1+|z|^2)^{p/2}\,(\hat{u}(0,z,\epsilon)-\hat{u}(0,z,0))|^2\,dz \to 0 \quad \text{as} \quad \epsilon\to 0 \tag{33}$$

for all t in J.

Lemma (4) together with condition (7) give the boundedness of $Y_1(t, z, 0)$; hence, by the Lebesgue dominated convergence theorem, we have

$$\int_{E^m} |(1+|z|^2)^{p/2}\,(Y_1(t,z,\epsilon)-Y_1(t,z,0))\,\hat{u}(0,z,0)|^2\,dz \to 0 \tag{34}$$

as $\epsilon \to 0$ for all t in J.

Again conditions (4) and (7) and Lemma (4) give with the dominated convergence theorem

$$\int_{E^m} |(1+|z|^2)^{p/2}\,Y_2(t,z,\epsilon)\,\delta_t\hat{u}(0,z,\epsilon)|^2\,dz \to 0 \tag{35}$$

as $\epsilon \to 0$, for all t in J.

Hence (33), (34), and (35) imply the following:

$$\| u(t,x,\epsilon) - u(t,x,0)\|_p \to 0 \quad \text{as} \quad \epsilon\to 0, \quad \text{for all } t \text{ in } J.$$

Theorem (2) is proved.

References

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